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On the stability of Sand Piles Model

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ABSTRACT

In this paper we study the stability of the Sand Piles Model where grains can be added from outside on random columns. We prove that the infinite set of all stable configurations have a lattice structure which is a sublattice of the Young lattice. Moreover, we give the formulae for the smallest and greatest times to reach stable configurations.

1. Introduction

The Sand Piles Model (SPM) was introduced in 1987 by Bak, Tang and Wiesenfeld as a sample model of the Self-organized criticality (SOC) phenomena [8]. The authors simulated the behaviour of a sand pile which builds up when sand is dropped on a line. A configuration is modelled as a sequence of columns consisting of cubic sand grains such that the height of columns is decreasing from left to right. In this model, a sand grain can fall down from a column to its right neighbours if the difference of height of the two columns is at least two.

This model is investigated in many works in physics, combinatorics and computer science [2,4,6,7,10]. In particular, the study of stable configuration (configuration on which no grain can fall down) is quite important because it gives information on the convergence of the model. On the other hand, in many dynamical systems, a small action on the initial configuration can cause a great change of the model, and consequently, an unanticipated stable configuration. Hence the study of stable configurations gives also a description of the behaviour of the model under actions.

Actually, by considering the initial configuration being a column of grains, Goles and Kiwi showed that the model reaches always the same stable configuration [4]. Then, Goles, Morvan and Phan proved the same property for an arbitrary initial configuration [5,6]. After that, Formenti and Masson gave an algorithm to compute the stable configuration reached from any given initial configuration [3].

On the other hand, Dhar, Cori, Rossin and Borgne [2,1,9] studied the set of stable configurations of a very related model, the Abelian Sand Piles Model. In their model, when a stable configuration is reached, grains can be added from outside, which implies evolutions of the model, and finally, another stable configuration is reached. The authors proved that the set of recurrent stable configurations has an Abelian group structure. Alternatively, regarding the addition of grains from outside, Latapy and Phan studied SPM with the addition of grains on the first column alone, and they showed the recursive structure of the space of configurations [7].

In this paper, we consider a more extended Sand Piles Model where outside grains are added on random columns. More precisely, each time the model reaches a stable configuration, one grain is added to a random column, and the model evolves to reach another stable configuration, and so on. We investigate the study of all such stable configurations.

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In Section 2, we give a formal definition of this model and prove that the set of all stable configurations has a lattice structure which is a sublattice of the well-known Young lattice.

Finally, in Section 3, we compute explicitly the smallest and greatest times to reach a stable configuration from the initial configuration, and the smallest and greatest times to reach a stable configuration from another stable configuration. These times illustrate the behaviour of the model under outside actions. The key idea of this computation is the introduction of the notion "energy". Indeed, for each configuration, we treat each of its grain by defining the energy of a grain being the greatest number of its possible moves.

2. Extended Sand Piles Model and its stable configurations

In this section, we first give the formal definition of the Extended Sand Piles Model. Then we study the lattice structure of the set of its stable configurations.

As well as in almost works of SPM, we represent configurations of this model by integer partitions. So let us first give some preliminary notions:

Definition 1. (i) A *partition* is an integer sequence $a = (a_1, a_2, \ldots, a_k)$ such that $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ (by convention, $a_j = 0$ for all $j > k$ and $a_0 = \infty$). We call a_i *part* of partition a; and k *length* of a, and write $l(a) = k$. We say that a is a partition of n, or n is the *weight* of a, and write $w(a) = n$, if $\sum_{i=1}^{i=k} a_i = n$.

(ii) A *smooth partition* is a partition such that all differences between two consecutive parts are at most 1.

(iv) *Young's lattice* is the lattice of all partitions ordered by containment [11] (*i.e.* $a \leq b$ if and only if $a_i \geq b_i$ for all $i = 1, 2, \ldots, \min\{l(a), l(b)\}$).

From this definition, one can see that a stable configuration is represented by a smooth partition. So, in the following, we say partition (resp. smooth partition) for configuration (resp. stable configuration).

The Extended Sand Piles Model (ESPM) is a discrete dynamical model where all configurations are partitions and the initial configuration being the partition (0). This model consists of two evolution (or transition) rules:

- *Falling rule (inside action)*: one grain on the column i can fall down to the column $i + 1$ if the height difference between the column i and the column $i + 1$ is greater than or equal to 2.
- *Adding rule (outside action)*: one grain can be added to one column of a smooth partition such that the obtained one is still a partition.

We denote also *ESPM* the set of all reachable partitions from the initial (0). On this set, we define the following relation: $a \geq b$ if b can be obtained from a by applying a sequence of transitions. One can easily check that this relation is an order relation. Besides, we call a *chain* in this model a sequence of transitions. By convention, a chain of one element (with no transitions) is of length 0. More particularly, a chain between two smooth partitions is called an *avalanche chain*. Finally, we denote by $a^{\downarrow i}$ the integer sequence obtained from a by increasing part i of a by 1.

Fig. 1 shows first elements of *ESPM*. One can see that *ESPM* does not contain all partitions. However, we prove in the following that this model contains all smooth partitions.

Proposition 1. *All smooth partitions are reachable from the initial partition.*

Proof. We prove this statement by recurrence on the total number of grains. Suppose that all smooth partitions of n are reachable from (0), we show that an arbitrary smooth partition $a = (a_1, \ldots, a_k)$ of $n + 1$ is also reachable from (0). Let j be the first index such that $a_{j-1} = a_j$ or $j = 1$ and that $a_j = a_{j+1} + 1$ (note that by convention $a_{k+1} = 0$, so such an index j exists). Let b be the partition of n such that $b_i = a_i$ for all $i \neq j$ and $b_j = a_j - 1$. It is clear that b is a smooth partition of n and a is obtained from b by applying the adding rule at position j. This implies that starting from the initial partition (0), one can obtain all smooth partitions. □

Let us recall that a smooth partition is a stable configuration under the falling rule, and it is fixed configuration if the model has not outside actions (adding transitions). In order to study the behaviour of the model under outside actions, we investigate the set of all stable configurations and the relations between them.

We denote the induced subposet of all smooth partitions of the poset *ESPM* by $(SESPM, \leq_S)$. We will describe the nature of order relation in *SESPM*.

First, we analyse the movement of a grain when it is added from outside to a stable configuration. So, let $a = (a_1, \ldots, a_k)$ be a smooth partition. One grain is added on column i of a with the condition that $a_i < a_{i-1}$. After that, if $a_i = a_{i+1}$, this grain stays at column i and does not move anymore. Otherwise, this grain move to a new position $j > i$ such that $a_i, a_{i+1}, \ldots, a_j$ is a consecutive decreasing integers and that $a_j = a_{j+1}$. Finally, this grain stays at column j and does not move anymore. The obtained configuration b of this sequence of moves of this grain is the same as the configuration obtained by only one move: adding a grain directly on position j. Hence, this analyse proves the following result:

Lemma 1. *In the SESPM, an element b is an immediate successor of an element a if and only if b can be obtained from a by adding one grain at some column.*

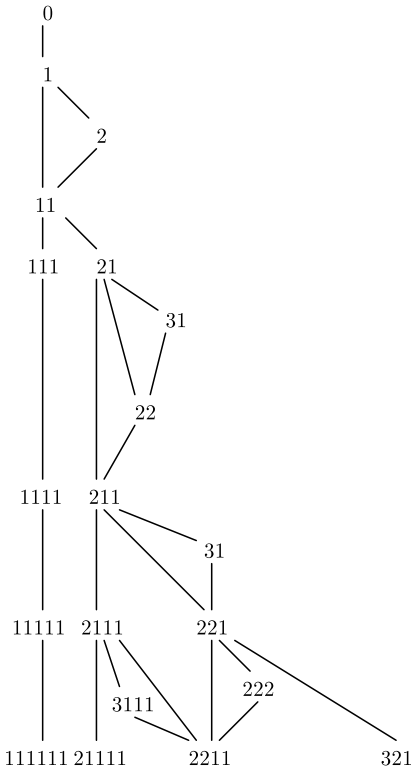

Fig. 1. First elements of the poset *ESPM*.

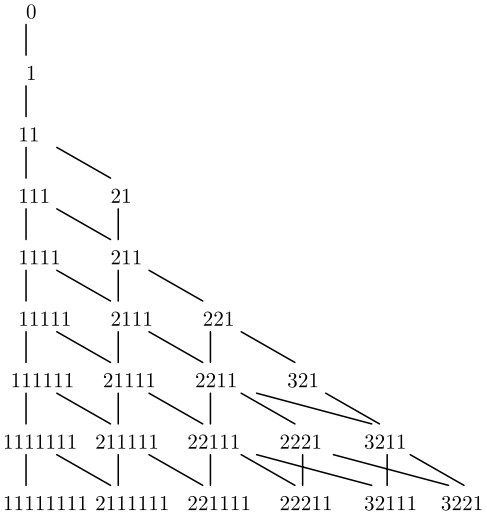

Fig. 2. First elements of the poset *SESPM*.

Therefore, the order relation $\leq_S$ can be well-understood by the following characterization:

Proposition 2. *The poset SESPM is ordered by containment.*

Fig. 2 shows some first elements of the poset *SESPM*. To finish this section, we discuss about the relation between *SESPM* and the Young lattice. Due to the characterization of the containment order, we know that the poset *SESPM* is a suborder of the Young lattice. Furthermore we prove that this relation is in fact a sublattice relation.

Theorem 1. *The poset SESPM is a sublattice of the Young lattice.*

Proof. We must to prove that for two arbitrary smooth partitions a and b, the two elements $\sup(a, b)$ and $\inf(a, b)$ in the Young lattice are also smooth partitions. Now, let $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_l)$. In the Young lattice, the

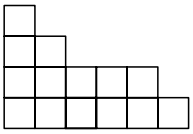

Fig. 3. The representation of the Ferrer diagram of the partition $a = (4, 3, 2, 2, 2, 1)$.

supremum of a and b is computed as follows:

$$\sup\{a, b\} = \big(\min\{a_1, b_1\}, \min\{a_2, b_2\}, \ldots, \min\{a_t, b_t\}\big),$$

where $t = \min\{k, l\}$.

We constate that for all $i = 1, 2, \ldots, t-1$, one has:

$$\min\{a_{i+1}, b_{i+1}\} \leq a_{i+1} \leq a_i + 1 \quad \text{and} \quad \min\{a_{i+1}, b_{i+1}\} \leq b_{i+1} \leq b_i + 1.$$

This implies that $\min\{a_{i+1}, b_{i+1}\} \leq \min\{a_i + 1, b_i + 1\} = 1 + \min\{a_i, b_i\}$. Which means that $\sup\{a, b\}$ is a smooth partition.

Similarly, the infimum of a and b in the Young lattice is computed by:

$$\inf\{a, b\} = \big(\max\{a_1, b_1\}, \max\{a_2, b_2\}, \ldots, \max\{a_h, b_h\}\big),$$

where $h = \max\{k, l\}$.

We have for all $i = 1, 2, \ldots, t-1$,

$$a_{i+1} \leq a_i + 1 \leq \max\{a_i + 1, b_i + 1\} \quad \text{and} \quad b_{i+1} \leq b_i + 1 \leq \max\{a_{i+1}, b_{i+1}\}.$$

Hence $\max\{a_{i+1}, b_{i+1}\} \leq \max\{a_i + 1, b_i + 1\} = 1 + \max\{a_i, b_i\}$. Which implies that $\inf\{a, b\}$ is a smooth partition. □

3. Avalanche chains

The purpose of this section is to describe the needed time to reach a stable configuration in the Extended Sand Piles Model. In the previous section, we know that there are probably different sequences of evolutions to reach a stable configuration from another stable configuration. Their sizes may be quite different and depend on the columns in which evolution rules are applied. We next show that the smallest length of avalanche chain depends only on the weight of the considered stable configurations. Otherwise, the problem is much more complicated than for the greatest length.

Theorem 2. *Let a and b be two smooth partitions and $b \leq_S a$. Then*
(i) *The smallest length of avalanche chain from the initial configuration* (0) *to a is equal to $w(a)$.*
(ii) *The smallest length of avalanche chain from a to b is equal to $w(b) - w(a)$.*

Proof. We prove the statement for (ii), the case (i) is a special case of (ii).

First, it is evident that the length of a chain from a to b is greater than or equal to $w(b) - w(a)$. So, our task is now to construct a chain of length $w(b) - w(a)$ from a to b. For that, it is sufficient to find a smooth partition b' of weight $w(b) - 1$ such that b is obtained directly from b' by a transition and b' is still smaller than a (by containment order). Let ℓ be the first index such that $b_\ell > a_\ell$, and let $j \geq l$ be the smallest index such that $b_{j-1} = b_j$ if $j > 1$ and that $b_j = b_{j+1} + 1$. Now, let $b' = (b_1, \ldots, b_{j-1}, b_j - 1, b_{j+1}, \ldots, b_k)$. It is easy to check that b is obtained from b' by adding one grain at column j and that b' satisfies our condition. □

To compute the greatest length of avalanche chains, we consider the movement of grains. We constate that, when one grain is added to a smooth partition, it slides down to a position until the obtained partition is smooth and after that this grain cannot be moved. Hence the number of moves of a grain depends only the position (column) where it is added. We will define by energy of a grain its greatest number of possible moves. Then we will define energy of a configuration the summation of energy of all of its grains. The main result of this section is to prove that the greatest length of avalanche chain to reach a stable configuration is equal to its energy.

Let us recall that, in our model, each configuration is represented by a partition, or more precisely, by its Ferrer diagram, where each grain is represented by a case (i, j) where i (resp. j) is the column (resp. row) index. So, let us denoted by $F(a)$ the diagram of a partition $a = (a_1, a_2, \ldots, a_k)$, and we write $(i, j) \in F(a)$ for all case (i, j) such that $1 \leq i \leq k$ and $1 \leq j \leq a_i$ (see Fig. 3 as an example). We say that i is a *smooth column* of a if $i = 1$ or $a_i = a_{i-1}$ for $i > 1$. Moreover, for a case (i, j), we define *diagonal* $D(i, j)$ the set of all case (i', j') such that $i' + j' = i + j$ and $1 \leq j' \leq j$ (see Fig. 4). We give now the formal definition and some properties of energy.

Definition 2. Let $a = (a_1, a_2, \ldots, a_k)$ be a smooth partition.
(i) The energy $e_a(i, j)$ is the greatest possible moves that a grain can do to reach the position (i, j).
(ii) The energy $E(a)$ of a is $E(a) = \sum_{(i,j)\in F(a)} e_a(i, j)$.

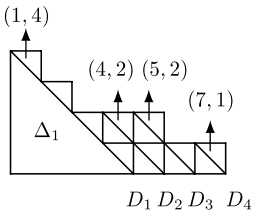

Fig. 4. Smooth columns (1, 4, 5, 7) and corresponding diagonals D_1, D_2, D_3, D_4 of $b = (4, 3, 2, 2, 2, 1, 1)$.

Lemma 2. *Let $a = (a_1, a_2, \ldots, a_k)$ be a smooth partition.*
(i) *We have: $e_a(i, j) = i + 1 - \min\{r : a_r < a_{r-1}$ and $a_r + r \geq j + i - 1\}$.*
(ii) *Moreover, if $(i, j) \in F(a)$ and $(i - 1, j + 1) \in F(a)$ then*

$$e_a(i - 1, j + 1) = e_a(i, j) - 1.$$

Proof. (i) We know that once a grain is added on a column of a smooth partition, it falls down diagonally from the head of the added column and it stops only when the diagonal is broken. Let r be a column index such that if a grain is added at column r, it can move to the position (i, j). Such an index r satisfies two following conditions: $a_r < a_{r-1}$, and $a_r + r \geq i + j - 1$. To have a greatest number of moves for the grain situated at position (i, j), we must find the smallest index $r_a(i, j)$ among all above column indices r, hence

$$r_a(i, j) = \min\{r : a_r < a_{r-1} \text{ and } a_r + r \geq j + i - 1\}.$$

Finally, if a grain is added to column r and moves to column i, then there is one adding transition and $i - r$ falling transitions. Then we have $e_a(i, j) = i + 1 - r_a(i, j)$.

(ii) The statement comes directly from the definition of $r_a(i, j)$. □

Now, we want to compute explicitly the energy of a smooth partition $a = (a_1, \ldots, a_k)$. Let $1 = i_1 < i_2 < \cdots < i_\ell$ be all smooth columns of a. And let D_i the diagonal (i, a_i). It is evident that we can decompose $F(a)$ as the following disjoint union:

$$F(a) = \Delta_1 \bigsqcup D_2 \bigsqcup \ldots \bigsqcup D_\ell$$

where Δ_1 is the set of all case (i, j) such that $1 \leq i, j$ and $i + j \leq a_1 + 1$. We then compute the energy of a in each of such subset.

Proposition 3. *Let a be a smooth partition, and let $1 = i_1 < i_2 < \cdots < i_\ell$ be all smooth columns of a. We have:*

$$E(a) = \frac{a_1(a_1 + 1)(a_1 + 2)}{6} + \sum_{r=2}^{\ell} i_r a_{i_r} - \sum_{r=3}^{\ell} i_{r-1} a_{i_r} + \sum_{r=2}^{\ell} \frac{a_{i_r}(a_{i_r} - 1)}{2}.$$

Proof. Let us analyse in each part of the decomposition of $F(a)$ (see Fig. 5).

- For Δ_1. It is clear that for each case (i, j) in Δ_1, we have $j \leq 1 + a_{i_1} - i$ and the energy $e_a(i, j)$ is equal to i, so we have:

$$\sum_{(i,j)\in\Delta_1} e_a(i, j) = \sum_{i=1}^{a_1} i(a_1 - i + 1) = \frac{a_1(a_1 + 1)(a_1 + 2)}{6}.$$

- For D_2. The energy of case (i_2, a_{i_2}) is equal to i_2 because if a grain is added on column 1, it can fall down to the case (i_2, a_{i_2}). Furthermore, due to (ii) of Lemma 2, in the diagonal D_2 from case (i_2, a_{i_2}) to case $(i_2 + a_{i_2} - 1, 1)$, the energy increases by 1, hence:

$$\sum_{(i,j)\in D_2} e_a(i, j) = \sum_{i=0}^{a_{i_2}-1} (i_2 + i) = i_2 a_{i_2} + \frac{a_{i_2}(a_{i_2} - 1)}{2}.$$

- For D_r with $3 \leq r \leq \ell$. We take care of the smallest column on which one can add a grain to move it to the position (i_r, a_{i_r}). By definition of smooth column, for all $s < i_{r-1}$, we have $a_s + s < a_{i_r} + i_r - 1$, so we cannot add a grain on column s. Regarding to column i_{r-1}, to move a grain to position (i, j), one can add a grain only at position $(i_{r-1}, a_{i_{r-1}} + 1)$. But at this moment, the height of column i_{r-1} is greater than the height of column $i_{r-1} - 1$ because of the smoothness of the column i_{r-1}, and this implies a contradiction. So we must to treat column s with $s \geq i_{r-1} + 1$. There are two case now:
  – if $i_r = i_{r-1} + 1$, so we must to add directly a grain on position i_r,
  – otherwise, we can add a grain on the column $i_{r-1} + 1$.
  This implies that, in any case, the desired column to add a grain is $i_{r-1} + 1$. So the energy of the case (i_r, a_{i_r}) is $i_r + 1 - (i_{r-1} + 1) = i_r - i_{r-1}$.

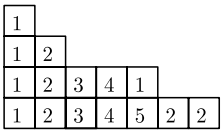

Fig. 5. Representation of the partition $b = (4, 3, 2, 2, 2, 1, 1)$, the number in each case is the energy of the corresponding grain. The greatest length from (0) to (b) is 34.

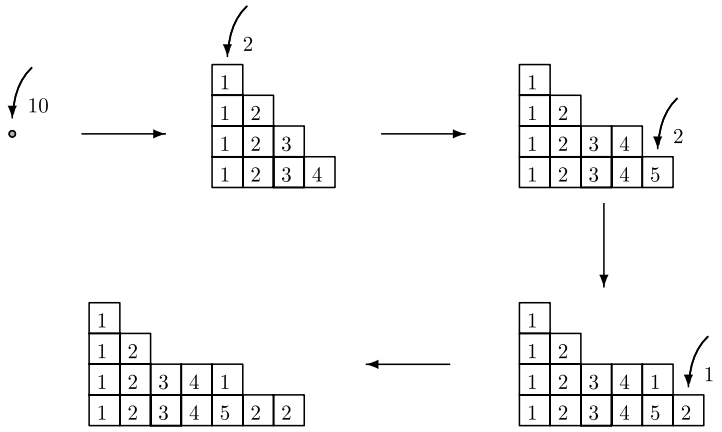

Fig. 6. A greatest chain from 0 to b. Each arrow $\to^k$ to a column i means that k grains are added to column i.

Using a similar argument like in the case of D_2, we obtain:

$$\sum_{(i,j)\in D_r} e_a(i,j) = \sum_{i=0}^{a_{i_r}-1} (i_r - i_{r-1} + i) = (i_r - i_{r-1})a_{i_r} + \frac{a_{i_r}(a_{i_r}-1)}{2}.$$

Finally, the total energy of a is given by:

$$\begin{aligned} E(a) &= \sum_{(i,j)\in\Delta_1} e_a(i,j) + \sum_{(i,j)\in D_2} e_a(i,j) + \sum_{r=3}^{\ell} \sum_{(i,j)\in D_r} e_a(i,j) \\ &= \frac{a_1(a_1+1)(a_1+2)}{6} + i_2 a_{i_2} + \frac{a_{i_2}(a_{i_2}-1)}{2} + \sum_{r=3}^{\ell} \left((i_r - i_{r-1})a_{i_r} + \frac{a_{i_r}(a_{i_r}-1)}{2} \right) \\ &= \frac{a_1(a_1+1)(a_1+2)}{6} + \sum_{r=2}^{\ell} i_r a_{i_r} - \sum_{r=3}^{\ell} i_{r-1} a_{i_r} + \sum_{r=2}^{\ell} \frac{a_{i_r}(a_{i_r}-1)}{2}. \quad \square \end{aligned}$$

We state now the main result of this section.

Theorem 3. *Let a be a smooth partition. Then the greatest length of avalanche chains from* (0) *to a is equal to $E(a)$.*

Proof. By statement (i) of Lemma 2, the length of an arbitrary avalanche chain from the initial partition (0) to a is less than or equal to $E(a)$. So our task is now to show an explicit way for adding grain from (0) to a such that the length of the induced avalanche chain is exactly equal to $E(a)$. For that, we fill consecutively each part of the decomposition of $F(a)$ (see Fig. 6).

- For Δ_1. By iterating $\frac{a_1(a_1+1)(a_1+2)}{6}$ adding grains always on column 1 we obtain Δ_1. Moreover, one can check that each grain situated in a case (i, j) in Δ_1 has exactly $e_a(i, j)$ moves.
- For D_2. By iterating a_{i_2} adding grains always on column 1, we obtain the configuration $\Delta_1 \bigsqcup D_2$. Moreover, each grain added will fall down along the diagonal D_2, then a grain situated in a case (i, j) in D_2 has exactly $e_a(i, j)$ moves.
- For D_r with $3 \le r \le \ell$. By iterating a_{i_r} adding grains always on column $a_{i_{r-1}+1}$, we obtain the configuration $\Delta_1 \bigsqcup D_2 \bigsqcup \ldots \bigsqcup D_r$. Similarly like in the case of D_2, each grain situated in a case (i, j) in D_r has exactly $e_a(i, j)$ moves.

Therefore, we constructed an avalanche chain from (0) to reach a where the number of moves of each grain is equal to the energy of this grain, so the length of this avalanche chain is equal to the energy of a. $\square$

From this theorem we can study avalanche chain between two stable configurations.

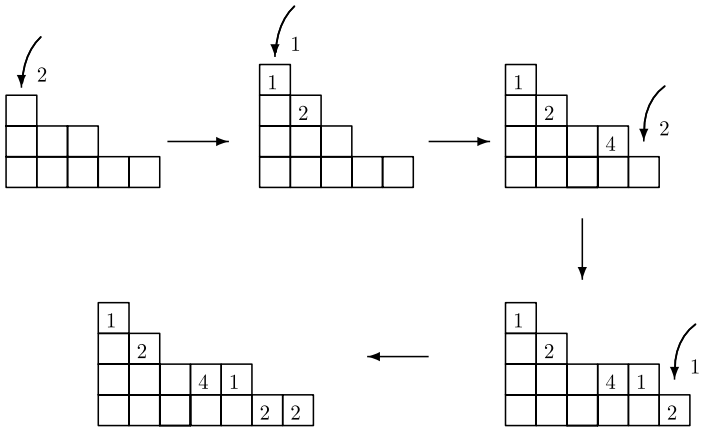

Fig. 7. A greatest chain from a to b. Each arrow $\rightarrow^k$ to a column i means that k grains are added to column i. The greatest length from a to b is 12.

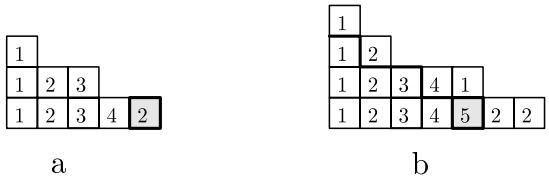

Fig. 8. (a) Energy tableau of a; (b) Energy tableau of b and $2 = e_a(5, 1) \neq e_b(5, 1) = 5$.

Corollary 1. *Let $b \leq_S a$ be two smooth partitions. Then the greatest length of avalanche chains from a to b is $\sum_{(i,j)\in F(b)-F(a)} e_b(i, j)$.*

Proof. The idea of this proof is similar to the one of Theorem 3. We treat each part of the decomposition of $F(b)$ and add missing grains to complete this part (see Fig. 7).

More precisely, the strategy is the follows. Let $n_i(a)$ (resp. $n_i(b)$) be the number of grains of a (resp. b) in the ith part of this decomposition. From the configuration a, we add always $n_1(b) - n_1(a)$ grains on column 1 to complete $\Delta_1(b)$. Then we add also $n_2(b) - n_2(a)$ grains on column 1 to complete $D_2(b)$. After that, for each $3 \leq r \leq \ell$, we add $n_r(b) - n_r(a)$ grains on column $i_{r-1} + 1$ to complete $D_r(b)$. One can check that the number of moves of each grain is always equal to its energy in b. This completes the proof. □

Nevertheless, it is important to note that the greatest length from a to b is not equal to $E(b) - E(a)$ because for $(i, j) \in F(b)$, we have not $e_a(i, j) = e_b(i, j)$ (see Fig. 8 for an encounter example). Moreover, it is easy to see that $e_b(i, j) \geq e_a(i, j)$. This implies that the greatest length of avalanche chains from a to b is smaller than or equal to the difference of the one from (0) to b and the one from (0) to a. This result is opposite to the result in the case of smallest length where the egality is hold.

Furthermore, we remark that the avalanche chain of greatest length from (0) to a is unique. Indeed, from the proof of Theorem 3, we constate that the grain G at position (i, j) on the diagonal D_r for $r \geq 2$ (resp. Δ_1) has exactly $e_a(i, j)$ transitions if and only if G is added at the column $i_{r-1}+1$ (resp. 1) and then it slides diagonally and stops at position (i, j). So the diagonal D_{r-1} must be fulfilled,moreover the grain at position $(i + 1, j - 1)$ must be presented before the adding of the grain G. So by recurrence we claim that the avalanche chain of greatest length from (0) to a must be defined explicitly as in the proof of Theorem 3, hence it is unique.

However, there are many avalanche chains of greatest length from a to b. For instance, if we take $a = (2, 2, 1, 1, 1)$ and $b = (2, 2, 2, 1, 1, 1)$. Then by Corollary 1 we have $l(a, b) = 2$. Moreover, we have the following two avalanche chains of length 2:

$$a = (2, 2, 1, 1, 1) \rightarrow (2, 2, 2, 1, 1) \rightarrow (2, 2, 2, 1, 1, 1) = b$$
$$\text{and} \quad a = (2, 2, 1, 1, 1) \rightarrow (2, 2, 1, 1, 1) \rightarrow (2, 2, 2, 1, 1, 1) = b.$$

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